

Shift techniques for Quasi-Birth and Death processes: canonical factorizations and matrix equations*

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Abstract

We revisit the shift technique applied to Quasi-Birth and Death (QBD) processes (He, Meini, Rhee, SIAM J. Matrix Anal. Appl., 2001) by bringing the attention to the existence and properties of canonical factorizations. To this regard, we prove new results concerning the solutions of the quadratic matrix equations associated with the QBD. These results find applications to the solution of the Poisson equation for QBDs.

Keywords: Quasi-Birth-and-Death processes, Shift technique, Canonical factorizations, Quadratic matrix equations

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1 Introduction

Quadratic matrix equations of the kind

$$A_{-1} + (A_0 - I)X + A_1X^2 = 0, \quad (1)$$

where A_{-1}, A_0, A_1 are given $n \times n$ matrices, are encountered in many applications, say in the solution of the quadratic eigenvalue problem, like vibration analysis, electric circuits, control theory and more [18, 13]. In the area of Markov chains, an important application concerns the solution of Quasi-Birth-and-Death (QBD) stochastic processes, where it is assumed that A_{-1} , A_0 and A_1 are nonnegative matrices such that $A_{-1} + A_0 + A_1$ is stochastic and irreducible [15, 3].

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For this class of problems, together with (1), the dual equation $X^2 A_{-1} + X(A_0 - I) + A_1 = 0$ has a relevant interest. It is well known that both (1) and the dual equation have minimal nonnegative solutions G and R , respectively, according to the component-wise ordering, which can be explicitly related to one another [15, 17]. These solutions have an interesting probabilistic interpretation and their computation is a fundamental task in the analysis of QBD processes. Moreover they provide the factorization $\varphi(z) = (I - zR)K(I - z^{-1}G)$ of the Laurent polynomial $\varphi(z) = z^{-1}A_{-1} + A_0 - I + zA_1$, where K is a nonsingular matrix. A factorization of this kind is canonical if $\rho(R) < 1$ and $\rho(G) < 1$, where ρ denotes the spectral radius. It is said weak canonical if $\rho(R) \leq 1$ and $\rho(G) \leq 1$.

We introduce the matrix polynomial $B(z) = A_{-1} + z(A_0 - I) + z^2 A_1 = z\varphi(z)$ and define the roots of $B(z)$ as the zeros of the polynomial $\det B(z)$. If ξ is a root of $B(z)$ we say that v is an eigenvector associated with ξ if $v \neq 0$ and $B(\xi)v = 0$. The location of the roots of $B(z)$ determines the classification of the QBD as positive, null recurrent or transient, and governs the convergence and the efficiency of the available numerical algorithms for approximating G and R [3]. In particular, $B(z)$ has always a root on the unit circle, namely, the root $\xi = 1$, and the corresponding eigenvector is the vector e of all ones, i.e., $B(1)e = 0$.

If the QBD is recurrent, the root $\xi = 1$ is the eigenvalue of largest modulus of the matrix G and $Ge = e$. In the transient case, that root is the eigenvalue of largest modulus of R . These facts have been used to improve convergence properties of numerical methods for computing the matrix G . The idea, introduced in [11] and based on the results of [5], is to “shift” the root $\xi = 1$ of $B(z)$ to zero or to infinity, and to construct a new quadratic matrix polynomial $B_s(z) = A_{-1}^s + z(A_0^s - I) + z^2 A_1^s$ having the same roots as $B(z)$, except for the root equal to 1, which is replaced with 0 or infinity. Here the super-(sub-)script s means “shifted”. This idea has been subsequently developed and applied in [4, 8, 9, 10, 14, 16].

In this paper we revisit the shift technique, and we focus on the properties of the canonical factorizations. In particular, we prove new results concerning the existence and properties of the solutions of the quadratic matrix equations obtained after the shift.

By following [3], we recall that in the positive recurrent case the root $\xi = 1$ can be shifted to zero by multiplying $B(z)$ to the right by a suitable function (right shift), while in the transient case the root $\xi = 1$ can be shifted to infinity by multiplying $B(z)$ to the left by another suitable function (left shift). In the null recurrent case, where $\xi = 1$ is a root of multiplicity 2, shift is applied both to the left and to the right so that one root 1 is shifted to zero and the other root 1 is shifted to infinity (double shift). In all the cases, the new Laurent matrix polynomial $\varphi_s(z) = z^{-1}B_s(z)$ is invertible on an annulus containing the unit circle in the complex plane and we prove that it admits a canonical factorization which is related to the weak canonical factorization of $\varphi(z)$. As a consequence, we relate G and R with the solutions G_s and R_s of minimal spectral radius of the matrix equations $A_{-1}^s + (A_0^s - I)X + A_1^s X^2 = 0$ and

$X^2 A_{-1}^s + X(A_0^s - I) + A_1^s = 0$, respectively.

A less trivial issue is the existence of the canonical factorization of $\varphi_s(z^{-1})$. We show that such factorization exists and we provide an explicit expression for it, for the three different kinds of shifts. The existence of such factorization allows us to express the minimal nonnegative solutions \hat{G} and \hat{R} of the matrix equations $A_{-1}X^2 + (A_0 - I)X + A_1 = 0$ and $A_{-1} + X(A_0 - I) + X^2 A_1 = 0$, in terms of the solutions of minimal spectral radius \hat{G}_s and \hat{R}_s of the equations $A_{-1}^s X^2 + (A_0^s - I)X + A_1^s = 0$ and $A_{-1}^s + X(A_0^s - I) + X^2 A_1^s = 0$, respectively.

The existence of the canonical factorizations of $\varphi_s(z)$ and $\varphi_s(z^{-1})$ has interesting consequences. Besides providing computational advantages in the numerical solution of matrix equations, it allows one to give an explicit expression for the solution of the Poisson problem for QBDs [2]. Another interesting issue related to the shift technique concerns conditioning. In fact, while null recurrent problems are ill-conditioned, the shifted counterparts are not. A convenient computational strategy to solve a null recurrent problem consists in transforming it into a new one, say by means of the double shift; solve the latter by using a quadratic convergent algorithm like cyclic reduction or logarithmic reduction [3]; then recover the solution of the original problem from the one of the shifted problem. For this conversion, the expressions relating the solutions of the shifted equations to those of the original equations are fundamental, they are provided in this paper.

The paper is organized as follows. In Section 2 we recall some properties of the canonical factorization of matrix polynomials, and their interplay with the solutions of the associated quadratic matrix equations, with specific attention to those equations encountered in QBD processes. In Section 3 we present the shift techniques in functional form, with attention to the properties of the roots of the original and modified matrix polynomial. In Section 4 we state the main results on the existence and properties of canonical factorizations. In particular we provide explicit relations between the solutions of the original matrix equations and the solutions of the shifted equations. In the Appendix, the reader can find the proof of a technical property used to prove the main results.

2 Preliminaries

In this section we recall some properties of matrix polynomials and of QBDs, that will be used later in the paper. For a general treatment on these topics we refer to the books [3, 7, 12, 15, 17].

2.1 Matrix polynomials

Consider the matrix Laurent polynomial $\varphi(z) = \sum_{i=-1}^1 z^i B_i$, where B_i , $i = -1, 0, 1$, are $n \times n$ complex matrices. A canonical factorization of $\varphi(z)$ is a decomposition of the kind $\varphi(z) = E(z)F(z^{-1})$, where $E(z) = E_0 + zE_1$ and $F(z) = F_0 + zF_{-1}$ are invertible for $|z| \leq 1$. A canonical factorization is *weak* if $E(z)$ and $F(z)$ are invertible for $|z| < 1$ but possibly singular for some values

of z such that $|z| = 1$. The canonical factorization is unique in the form $\varphi(z) = (I - z\tilde{E}_1)K(I - z^{-1}\tilde{F}_{-1})$ for suitable matrices \tilde{E}_1 , \tilde{F}_{-1} and K , see for instance [6].

Given an $n \times n$ quadratic matrix polynomial $B(z) = B_{-1} + zB_0 + z^2B_1$, we call *roots of $B(z)$* the roots ξ_1, \dots, ξ_{2n} of the polynomial $\det B(z)$ where we assume that there are k roots at infinity if the degree of $\det B(z)$ is $2n - k$. In the sequel we also assume that the roots are ordered so that $|\xi_1| \leq \dots \leq |\xi_{2n}|$.

Consider the following matrix equations

$$B_{-1} + B_0X + B_1X^2 = 0, \quad (2)$$

$$X^2B_{-1} + XB_0 + B_1 = 0, \quad (3)$$

$$B_{-1}X^2 + B_0X + B_1 = 0, \quad (4)$$

$$B_{-1} + XB_0 + X^2B_1 = 0. \quad (5)$$

Observe that if X is a solution of (2) and $Xv = \lambda v$ for some $v \neq 0$, then $B(\lambda)v = 0$ that is, λ is a root of $B(z)$. Similarly, the eigenvalues of any solution of (5) are roots of $B(z)$, and the reciprocal of the eigenvalues of any solution of (3) or (4) are roots of $B(z)$ as well. Here we adopt the convention that $1/0 = \infty$ and $1/\infty = 0$.

We state the following general result on canonical factorizations which extends Theorem 3.20 of [3]:

Theorem 1. *Let $\varphi(z) = z^{-1}B_{-1} + B_0 + zB_1$ be an $n \times n$ Laurent matrix polynomial. Assume that the roots of $B(z) = z\varphi(z)$ are such that $|\xi_n| < 1 < |\xi_{n+1}|$ and that there exists a matrix G which solves the matrix equation (2) with $\rho(G) = |\xi_n|$. Then the following properties hold:*

1. $\varphi(z)$ has the canonical factorization $\varphi(z) = (I - zR)K(I - z^{-1}G)$, where $K = B_0 + B_1G$, $R = -B_1K^{-1}$, $\rho(R) = 1/|\xi_{n+1}|$ and R is the solution of the equation (3) with minimal spectral radius.
2. $\varphi(z)$ is invertible in the annulus $\mathbb{A} = \{z \in \mathbb{C} : |\xi_n| < z < |\xi_{n+1}|\}$ and $H(z) = \varphi(z)^{-1} = \sum_{i=-\infty}^{+\infty} z^i H_i$ is convergent for $z \in \mathbb{A}$, where

$$H_i = \begin{cases} G^{-i}H_0, & \text{for } i < 0, \\ \sum_{j=0}^{+\infty} G^j K^{-1} R^j, & \text{for } i = 0, \\ H_0 R^i, & \text{for } i > 0. \end{cases}$$

3. If H_0 is nonsingular, then $\varphi(z^{-1})$ has the canonical factorization $\varphi(z^{-1}) = (I - z\hat{R})\hat{K}(I - z^{-1}\hat{G})$, where $\hat{K} = B_0 + B_1\hat{G} = B_0 + \hat{R}B_{-1}$ and $\hat{G} = H_0 R H_0^{-1}$, $\hat{R} = H_0^{-1} G H_0$. Moreover, \hat{G} and \hat{R} are the solutions of minimal spectral radius of the equations (4) and (5), respectively.

Proof. Parts 1 and 2 are stated in Theorem 3.20 of [3]. We prove Part 3. From 2, the function $H(z^{-1}) = \varphi(z^{-1})^{-1}$ is analytic in an annulus $\hat{\mathbb{A}}$ contained in \mathbb{A} and containing the unit circle. From the expression of H_i we obtain $H(z^{-1}) =$

$\sum_{i=1}^{+\infty} z^{-i} H_0 R^i + H_0 + \sum_{i=1}^{+\infty} z^i G^i H_0$. Since $\det H_0 \neq 0$, we may rewrite the latter equation as

$$\begin{aligned} H(z^{-1}) &= \sum_{i=1}^{+\infty} z^{-i} (H_0 R^i H_0^{-1}) H_0 + H_0 + \sum_{i=1}^{+\infty} z^i H_0 (H_0^{-1} G^i H_0) \\ &= \sum_{i=1}^{+\infty} z^{-i} \widehat{G}^i H_0 + H_0 + \sum_{i=1}^{+\infty} z^i H_0 \widehat{R}^i, \end{aligned}$$

where we have set $\widehat{G} = H_0 R H_0^{-1}$ and $\widehat{R} = H_0^{-1} G H_0$. Since the matrix power series in the above equation are convergent in $\widehat{\mathbb{A}}$, more precisely $\sum_{i=1}^{+\infty} z^{-i} \widehat{G}^i = (I - z^{-1} \widehat{G})^{-1} - I$ and $\sum_{i=1}^{+\infty} z^i \widehat{R}^i = (I - z \widehat{R})^{-1} - I$, we may write

$$\begin{aligned} H(z^{-1}) &= ((I - z^{-1} \widehat{G})^{-1} - I) H_0 + H_0 + H_0 ((I - z \widehat{R})^{-1} - I) \\ &= (I - z^{-1} \widehat{G})^{-1} Y (I - z \widehat{R})^{-1} \end{aligned}$$

where

$$Y = H_0 (I - z \widehat{R}) - (I - z^{-1} \widehat{G}) H_0 (I - z \widehat{R}) + (I - z^{-1} \widehat{G}) H_0 = H_0 - \widehat{G} H_0 \widehat{R}.$$

The matrix Y cannot be singular since otherwise $\det H(z^{-1}) = 0$ for any $z \in \widehat{\mathbb{A}}$, which contradicts the invertibility of $H(z^{-1})$. Therefore, we find that $\varphi(z^{-1}) = (I - z \widehat{R}) Y^{-1} (I - z^{-1} \widehat{G})$ for $z \in \widehat{\mathbb{A}}$, in particular for $|z| = 1$. This factorization is canonical since $\rho(\widehat{R}) = \rho(G) < 1$ and $\rho(\widehat{G}) = \rho(R) < 1$. By the uniqueness of canonical factorizations [6], one has $Y^{-1} = \widehat{K} = B_0 + B_{-1} \widehat{G}$. One finds, by direct inspection, that the matrices \widehat{G} and \widehat{R} are solutions of (4) and (5), respectively. Moreover, they are solutions of minimal spectral radius since their eigenvalues coincide with the n roots with smallest modulus of $B(z)$ and of $zB(z^{-1})$, respectively. \square

The following result holds under weaker assumptions and provides the converse property of part 3 of Theorem 1.

Theorem 2. *Let $\varphi(z) = z^{-1} B_{-1} + B_0 + z B_1$ be an $n \times n$ Laurent matrix polynomial such that the roots of $B(z) = z \varphi(z)$ satisfy $|\xi_n| \leq 1 \leq |\xi_{n+1}|$. The following properties hold:*

1. *If there exists a solution G to the matrix equations (2) such that $\rho(G) = |\xi_n|$, then $\varphi(z)$ has the (weak) canonical factorization $\varphi(z) = (I - zR)K(I - z^{-1}G)$, where $K = B_0 + B_1 G = B_0 + R B_{-1}$, $R = -B_1 K^{-1}$, and R is a solution of (3) with $\rho(R) = 1/|\xi_{n+1}|$;*
2. *if there exists a solution \widehat{G} to the matrix equation (4) such that $\rho(\widehat{G}) = 1/|\xi_{n+1}|$, then $\varphi(z^{-1})$ has the (weak) canonical factorization $\varphi(z^{-1}) = (I - z \widehat{R}) \widehat{K} (I - z^{-1} \widehat{G})$, where $\widehat{K} = B_0 + B_1 \widehat{G} = B_0 + \widehat{R} B_{-1}$, and \widehat{R} is a solution of (5) with $\rho(\widehat{R}) = |\xi_n|$;*

3. if $|\xi_n| < |\xi_{n+1}|$, and if there exist solutions G and \hat{G} to the matrix equations (2) and (4), respectively, such that $\rho(G) = |\xi_n|$, $\rho(\hat{G}) = |1/\xi_{n+1}|$, then the series $W = \sum_{i=0}^{\infty} G^i K^{-1} R^i$ is convergent, W is the unique solution of the Stein equation $X - GXR = K^{-1}$, W is nonsingular and $\hat{G} = WRW^{-1}$, $\hat{R} = W^{-1}GW$. Moreover, $W^{-1} = K(I - G\hat{G})$ and $I - G\hat{G}$ is invertible.

Proof. Properties 1 and 2 can be proved as Property 1 in Theorem 3.20 of [3]. Assume that $|\xi_n| < |\xi_{n+1}|$. Since $\rho(G)$ or $\rho(R)$ is less than one, the series $\sum_{i=0}^{\infty} G^i K^{-1} R^i$ is convergent. Observe that $GW R = \sum_{i=0}^{\infty} G^i K^{-1} R^i$ so that $W - GW R = K^{-1}$. Therefore W solves the Stein equation $X - GXR = K^{-1}$. The solution is unique since X solves the Stein equation if and only if $(I \otimes I - R^T \otimes G) \text{vec}(X) = \text{vec}(K^{-1})$, where $\text{vec}(\cdot)$ is the operator that stacks the columns of a matrix and \otimes is the Kronecker product; the matrix of the latter system is nonsingular since $\rho(R^T \otimes G) = \rho(G)\rho(R) < 1$. We prove that $\det(W) \neq 0$. Assume that $|\xi_n| \leq 1$ and $|\xi_{n+1}| > 1$ and choose $t \in \mathbb{R}$ such that $|\xi_n| < t < |\xi_{n+1}|$. Consider the matrix polynomial

$$B_t(z) := B(tz) = B_{-1,t} + zB_{0,t} + z^2B_{1,t},$$

where $B_{-1,t} = B_{-1}$, $B_{0,t} = tB_0$ and $B_{1,t} = t^2B_1$. The roots of $B_t(z)$ are $\xi_{i,t} = \xi_i/t$, $i = 1, \dots, 2n$. Therefore, for the chosen t , we have $|\xi_{n,t}| < 1 < |\xi_{n+1,t}|$. Moreover the matrices $G_t = t^{-1}G$ and $\hat{G}_t = t\hat{G}$ are solutions with spectral radius less than one of the matrix equations $B_{-1,t} + B_{0,t}X + B_{1,t}X^2 = 0$ and $B_{-1,t}X^2 + B_{0,t}X + B_{1,t} = 0$, respectively. In this way, the matrix polynomial $B_t(z)$ satisfies the assumptions of Theorem 3.20 of [3], and the matrix $H_{0,t} = \sum_{i=0}^{\infty} G_t^i (B_{0,t} + B_{1,t}G_t)^{-1} R_t^i$, where $R_t = tR$, is nonsingular. One verifies by direct inspection that $W = tH_{0,t}$. Therefore we conclude that W is nonsingular as well. Applying again Theorem 3.20 yields $\hat{G}_t = H_{0,t}R_tH_{0,t}^{-1}$ and $\hat{R}_t = H_{0,t}^{-1}G_tH_{0,t}$, where $\hat{R}_t = t^{-1}\hat{R}$, therefore $\hat{G} = WRW^{-1}$ and $\hat{R} = W^{-1}GW$. Similar arguments may be used if $|\xi_n| < 1$ and $|\xi_{n+1}| \geq 1$. Concerning the expression of W^{-1} , by the definition of W we have $(I - G\hat{G})W = W - \sum_{j=0}^{\infty} G^{j+1}K^{-1}R^{j+1} = K^{-1}$, so that $W^{-1} = K(I - G\hat{G})$. \square

2.2 Nonnegative matrices, quadratic matrix equations and QBDs

A real matrix A is nonnegative (positive) if all its entries are nonnegative (positive), and we write $A \geq 0$ ($A > 0$). If A and B are real matrices, we write $A \geq B$ ($A > B$) if $A - B \geq 0$ ($A - B > 0$). An $n \times n$ real matrix $M = \alpha I - N$ is called an M-matrix if $N \geq 0$ and $\alpha \geq \rho(N)$. A useful property is that the inverse of a nonsingular M-matrix is nonnegative. For more properties on nonnegative matrices and M-matrices we refer to the book [1].

Assume we are given $n \times n$ nonnegative matrices A_{-1} , A_0 and A_1 such that $A_{-1} + A_0 + A_1$ is stochastic. The matrices A_{-1} , A_0 and A_1 define the

homogeneous part of the infinite transition matrix

$$P = \begin{bmatrix} A'_0 & A'_1 & 0 \\ A_{-1} & A_0 & A_1 \\ 0 & \ddots & \ddots & \ddots \end{bmatrix}$$

of a QBD with space state $\mathbb{N} \times \mathcal{S}$, $\mathcal{S} = \{1, \dots, n\}$, where A'_0 and A'_1 are $n \times n$ matrices [15]. We assume that the matrix P is irreducible and that the following properties are satisfied, they are not restrictive for models of practical interest:

Assumption 3. *The matrix $A_{-1} + A_0 + A_1$ is irreducible.*

Assumption 4. *The doubly infinite QBD on $\mathbb{Z} \times \mathcal{S}$ has only one final class $\mathbb{Z} \times \mathcal{S}_*$, where $\mathcal{S}_* \subseteq \mathcal{S}$. Every other state is on a path to the final class. Moreover, the set \mathcal{S}_* is not empty.*

Assumption 4 is Condition 5.2 in [3, Page 111] where it is implicitly assumed that \mathcal{S}_* is not empty.

We denote by G , R , \widehat{G} and \widehat{R} the minimal nonnegative solutions of the following equations

$$\begin{aligned} A_{-1} + (A_0 - I)X + A_1X^2 &= 0, \\ X^2A_{-1} + X(A_0 - I) + A_1 &= 0, \\ A_{-1}X^2 + (A_0 - I)X + A_1 &= 0, \\ A_{-1} + X(A_0 - I) + X^2A_1 &= 0, \end{aligned} \tag{6}$$

respectively. The matrices G , R , \widehat{G} and \widehat{R} exist, are unique and have a probabilistic interpretation [15]. If S is any of G , R , \widehat{G} and \widehat{R} , we denote by ρ_S the spectral radius $\rho(S)$ of S , and we denote by u_S and v_S^T a nonnegative right and left Perron eigenvector of S , respectively, so that $Su_S = \rho_S u_S$ and $v_S^T S = \rho_S v_S^T$.

Define the matrix polynomial

$$B(z) = A_{-1} + z(A_0 - I) + z^2A_1 = B_{-1} + zB_0 + z^2B_1,$$

and the Laurent matrix polynomial $\varphi(z) = z^{-1}B(z)$. Denote by ξ_1, \dots, ξ_{2n} the roots of $B(z)$, ordered such that $|\xi_1| \leq \dots \leq |\xi_{2n}|$. According to Theorem 2 and [3, Theorem 5.20], the eigenvalues of G , R , \widehat{G} and \widehat{R} are related as follows to the roots ξ_i , $i = 1, \dots, 2n$.

Theorem 5. *The eigenvalues of G and \widehat{R} are ξ_1, \dots, ξ_n , while the eigenvalues of R and \widehat{G} are $\xi_{n+1}^{-1}, \dots, 1/\xi_{2n}^{-1}$. Moreover ξ_n, ξ_{n+1} are real positive, and:*

1. *if the QBD is positive recurrent then $\xi_n = 1 < \xi_{n+1}$, G is stochastic and \widehat{G} is substochastic;*
2. *if the QBD is null recurrent then $\xi_n = 1 = \xi_{n+1}$, G and \widehat{G} are stochastic;*
3. *if the QBD is transient then $\xi_n < 1 = \xi_{n+1}$, G is substochastic and \widehat{G} is stochastic.*

As a consequence of the above theorem we have $\rho_G = \rho_{\hat{R}} = \xi_n > 0$, $\rho_R = \rho_{\hat{G}} = \xi_{n+1}^{-1} > 0$. This way, since $A(z) = A_{-1} + zA_0 + z^2A_1$ is irreducible and nonnegative for $z > 0$, we find that u_G , $u_{\hat{G}}$ are the positive Perron vectors of $A(\xi_n)$ and $A(\xi_{n+1}^{-1})$, respectively. Similarly, v_R and $v_{\hat{R}}$ are positive left Perron vectors of $A(\xi_{n+1}^{-1})$ and $A(\xi_n)$, respectively. Moreover, if $\xi_n = \xi_{n+1}$, then $u_G = u_{\hat{G}} = e$, where e is the vector of all ones, and $v_R = v_{\hat{R}}$ with $v_R^T(A_{-1} + A_0 + A_1) = v_R^T$. Under Assumption 4, according to [3, Section 4.7], it follows that 1 is the only root of $B(z)$ of modulus 1.

Since G and R solve the first two equations in (6), we find that

$$\varphi(z) = (I - zR)K(I - z^{-1}G), \quad K = A_0 - I + A_1G = A_0 - I + RA_{-1}. \quad (7)$$

Similarly, since \hat{G} and \hat{R} solve the last two equations in (6), we have

$$\varphi(z^{-1}) = (I - z\hat{R})\hat{K}(I - z^{-1}\hat{G}), \quad \hat{K} = A_0 - I + A_{-1}\hat{G} = A_0 - I + \hat{R}A_1. \quad (8)$$

In view of Theorem 5 the decompositions (7) and (8) are weak canonical factorizations of $\varphi(z)$ and $\varphi(z^{-1})$, respectively. From (7) and (8) we have

$$\begin{aligned} A_1 &= -RK = -\hat{K}\hat{G}, & A_{-1} &= -KG = -\hat{R}\hat{K}, \\ A_1G &= RA_{-1}, & A_{-1}\hat{G} &= \hat{R}A_1. \end{aligned} \quad (9)$$

The following result provides some properties of the matrices involved in the above equations.

Theorem 6. *The following properties hold:*

1. $-K$ and $-\hat{K}$ are nonsingular M-matrices;
2. $v_G^TK^{-1}u_R < 0$ and $v_{\hat{G}}^T\hat{K}^{-1}u_{\hat{R}} < 0$;
3. the series $\sum_{i=0}^{\infty} G^iK^{-1}R^i$ and $\sum_{i=0}^{\infty} \hat{G}^i\hat{K}^{-1}\hat{R}^i$ are convergent if and only if the QBD is not null recurrent.

Proof. The matrix $U = A_0 + A_1G$ is nonnegative and

$$Uu_G = (A_0 + A_1G)u_G = \rho_G^{-1}(A_0G + A_1G^2)u_G.$$

Since G solves the equation (1), then $Uu_G = \rho_G^{-1}(G - A_{-1})u_G \leq u_G$. Since $u_G > 0$, this latter inequality implies that $\rho(U) \leq 1$; moreover, $\rho(U)$ cannot be one otherwise K would be singular, and from (7) the polynomial $\det B(z)$ would be identically zero. Hence, $-K$ is a nonsingular M-matrix. Similarly, $-\hat{K}$ is a nonsingular M-matrix. The proof of part 2 is rather technical and is reported in the Appendix. Concerning part 3, consider the series $\sum_{i=0}^{\infty} G^iK^{-1}R^i$. Since the matrix $-K$ is a nonsingular M-matrix, one has $K^{-1} \leq 0$, and the series has nonpositive terms since $G \geq 0$ and $R \geq 0$. In the null recurrent case $\rho(R) = \rho(G) = 1$, therefore the series diverges since $v_G^TK^{-1}u_R < 0$. In the other cases, the powers of G and R are uniformly bounded, and one of the matrices G and R has spectral radius less than one, therefore the series is convergent. \square

In the non null recurrent case, the matrices G and \widehat{R} on the one hand, \widehat{G} and R on the other hand, are related through the series $W = \sum_{i=0}^{\infty} G^i K^{-1} R^i$ as indicated by Part 3 of Theorem 2.

3 Shifting techniques for QBDs

The shift technique presented in this paper may be seen as an extension, to matrix polynomials, of the following result due to Brauer [5]:

Theorem 7. *Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Let x_k be an eigenvector of A associated with the eigenvalue λ_k , $1 \leq k \leq n$, and let q be any n -dimensional vector. The matrix $A + x_k q^T$ has eigenvalues $\lambda_1, \dots, \lambda_{k-1}, \lambda_k + x_k^T q, \lambda_{k+1}, \dots, \lambda_n$.*

The matrix polynomial $B(z) = A_{-1} + z(A_0 - I) + z^2 A_1$ has always a root on the unit circle, namely $z = 1$. This implies that $\varphi(z) = z^{-1} B(z)$ is not invertible on the unit circle and has only a weak canonical factorization (see formulas (7) and (8)). In this section we revisit in functional form the shift technique introduced in [11]. Starting from $\varphi(z)$ we construct a new Laurent matrix polynomial $\varphi_s(z)$ such that the roots of $B_s(z) = z\varphi_s(z)$ coincide with the roots of $B(z)$ except for one root, which is shifted away to zero or to infinity. Therefore we may apply this technique to remove the singularities on the unit circle. This can be performed in two different ways: by operating to the right of $\varphi(z)$ or operating to the left. We treat separately the two cases.

3.1 Shift to the right

Our aim in this section is to shift the root ξ_n of $B(z)$ to zero. To this end, we multiply $\varphi(z)$ on the right by a suitable matrix function.

Take $Q = u_G v^T$, where v is any vector such that $u_G^T v = 1$. Define

$$\varphi_r(z) = \varphi(z) \left(I + \frac{\xi_n}{z - \xi_n} Q \right). \quad (10)$$

where the suffix r denotes shift to the Right. We prove the following:

Theorem 8. *The function $\varphi_r(z)$ defined in (10) coincides with the Laurent matrix polynomial $\varphi_r(z) = z^{-1} A_{-1}^r + A_0^r - I + z A_1^r$ with matrix coefficients*

$$A_{-1}^r = A_{-1}(I - Q), \quad A_0^r = A_0 + \xi_n A_1 Q, \quad A_1^r = A_1. \quad (11)$$

Moreover, the roots of $B_r(z) = z\varphi_r(z)$ are $0, \xi_1, \dots, \xi_{n-1}, \xi_{n+1}, \dots, \xi_{2n}$.

Proof. Since $\xi_n = \rho_G$ and $B(\xi_n)u_G = 0$, then $A_{-1}Q = -\xi_n(A_0 - I)Q - \xi_n^2 A_1 Q$, and we have

$$\begin{aligned} B(z)Q &= -\xi_n(A_0 - I)Q - \xi_n^2 A_1 Q + (A_0 - I)Qz + A_1 Qz^2 \\ &= (z^2 - \xi_n^2)A_1 Q + (z - \xi_n)(A_0 - I)Q. \end{aligned}$$

This way we find that $\frac{\xi_n}{z-\xi_n}B(z)Q = \xi_n(z + \xi_n)A_1Q + \xi_n(A_0 - I)Q$, therefore

$$\varphi_{\mathbf{r}}(z) = \varphi(z) + \frac{\xi_n}{z-\xi_n}\varphi(z)Q = z^{-1}A_{-1}^{\mathbf{r}} + A_0^{\mathbf{r}} - I + A_1^{\mathbf{r}}z$$

so that (11) follows. As $\det(I + \frac{\xi_n}{z-\xi_n}Q) = \frac{z}{z-\xi_n}$, we have from (10) that $\det B_{\mathbf{r}}(z) = \frac{z}{z-\xi_n} \det B(z)$. This means that the roots of the polynomial $\det B_{\mathbf{r}}(z)$ coincide with the roots of $\det B(z)$ except for ξ_n which is replaced with 0. \square

We analyze the consequences of the above theorem. In the positive recurrent case, where $\xi_n = 1 < \xi_{n+1}$, the matrix polynomial $B_{\mathbf{r}}(z)$ has n roots of modulus strictly less than 1, and n of modulus strictly greater than 1; in particular, $B_{\mathbf{r}}(z)$ is nonsingular on the unit circle and on the annulus $|\xi_{n-1}| < |z| < \xi_{n+1}$. In the null recurrent case, where $\xi_n = 1 = \xi_{n+1}$, the matrix polynomial $B_{\mathbf{r}}(z)$ has n roots of modulus strictly less than 1, and n of modulus greater than or equal to 1; in particular, $B_{\mathbf{r}}(z)$ has a simple root at $z = 1$. In the transient case, where $\xi_n < 1 = \xi_{n+1}$, the splitting of the roots with respect to the unit circle is not changed, since $B_{\mathbf{r}}(z)$ has, like $B(z)$, n roots of modulus strictly less than 1, and n of modulus greater than or equal to 1.

It is worth pointing out that in the recurrent case the vector u_G is the vector of all ones and $\xi_n = 1$, therefore the quantities involved in the construction of the matrix polynomial $B_{\mathbf{r}}(z)$ are known *a priori*. In the transient case it is convenient to apply the shift to the root ξ_{n+1} , by moving it away to infinity. This is obtained by acting on $\varphi(z)$ to the left, as described in the next section.

3.2 Shift to the left

Consider the matrix $S = wv_R^T$, where w is any vector such that $v_R^T w = 1$. Define the matrix function

$$\varphi_{\ell}(z) = \left(I - \frac{z}{z - \xi_{n+1}} S \right) \varphi(z), \quad (12)$$

where the suffix ℓ denotes shift to the left.

Theorem 9. *The function $\varphi_{\ell}(z)$ defined in (12) coincides with the Laurent matrix polynomial $\varphi_{\ell}(z) = z^{-1}A_{-1}^{\ell} + A_0^{\ell} - I + zA_1^{\ell}$ with matrix coefficients $A_{-1}^{\ell} = A_{-1}$, $A_0^{\ell} = A_0 + \xi_{n+1}^{-1}SA_{-1}$, $A_1^{\ell} = (I - S)A_1$. Moreover, the roots of $B_{\ell}(z) = z\varphi_{\ell}(z)$ are $\xi_1, \dots, \xi_n, \xi_{n+2}, \dots, \xi_{2n}, \infty$.*

Proof. Since $\xi_{n+1} = \rho_R^{-1}$ and $v_R^T B(\rho_R^{-1}) = 0$, then $SA_{-1} = -\xi_{n+1}S(A_0 - I) - \xi_{n+1}^2 SA_1$, and we have

$$\begin{aligned} SB(z) &= -\xi_{n+1}S(A_0 - I) - \xi_{n+1}^2 SA_1 + S(A_0 - I)z + SA_1 z^2 \\ &= (z^2 - \xi_{n+1}^2)SA_1 + (z - \xi_{n+1})S(A_0 - I). \end{aligned}$$

This way we find that $\frac{z}{z-\xi_{n+1}}S\varphi(z) = (z + \xi_{n+1})SA_1 + S(A_0 - I)$, therefore

$$\varphi_{\ell}(z) = \varphi(z) - \frac{z}{z - \xi_{n+1}}S\varphi(z) = z^{-1}A_{-1}^{\ell} + A_0^{\ell} - I + zA_1^{\ell}$$

with

$$A_{-1}^\ell = A_{-1}, \quad A_0^\ell = A_0 + \xi_{n+1}^{-1} S A_{-1}, \quad A_1^\ell = (I - S) A_1.$$

As $\det(I - \frac{z}{z - \xi_{n+1}} S) = -\frac{1}{z - \xi_{n+1}}$, we have $\det B_\ell(z) = -\frac{1}{z - \xi_{n+1}} \det B(z)$ from (12). This means that the roots of the matrix polynomial $B_\ell(z)$ coincide with the roots of $B(z)$ except the root equal to ξ_{n+1} which has been moved to infinity. \square

A consequence of the above theorem is that in the transient case, when $\xi_n < 1 = \xi_{n+1}$, the matrix polynomial $B_\ell(z)$ has n roots of modulus strictly less than 1 and n roots of modulus strictly greater than 1 (included the root(s) at the infinity). In particular, $\varphi_\ell(z)$ is invertible on the unit circle and on the annulus $\xi_n < |z| < |\xi_{n+2}|$.

The shift to the left applied to the function $\varphi(z)$ in order to move the root ξ_{n+1} to the infinity, can be viewed as a shift to the right applied to the function $\tilde{\varphi}(z) = \varphi^T(z^{-1})$ to move the root ξ_{n+1}^{-1} to zero. In fact, observe that the roots of $z\tilde{\varphi}(z)$ are the reciprocals of the roots of $B(z)$ so that the roots ξ_n and ξ_{n+1} of $B(z)$ play the role of the roots ξ_{n+1}^{-1} and ξ_n^{-1} of $z\tilde{\varphi}(z)$ respectively. From (10) we have

$$\tilde{\varphi}_r(z) = \tilde{\varphi}(z) \left(I + \frac{\xi_{n+1}^{-1}}{z - \xi_{n+1}^{-1}} Q' \right)$$

for $Q' = v_R w^T$. Taking the transpose in both sides of the above equation yields

$$\tilde{\varphi}_r^T(z) = \left(I + \frac{\xi_{n+1}^{-1}}{z - \xi_{n+1}^{-1}} Q'^T \right) \tilde{\varphi}(z)^T.$$

Replacing z with z^{-1} yields (12) where $\varphi_\ell(z) = \tilde{\varphi}_r^T(z^{-1})$.

3.3 Double shift

The right and left shifts presented in the previous sections can be combined, yielding the double shift technique, where the new quadratic matrix polynomial $B_d(z)$ has the same roots of $B(z)$, except for ξ_n and ξ_{n+1} , which are shifted to 0 and to infinity, respectively.

By following the same arguments used in the previous sections, we define the matrix function

$$\varphi_d(z) = \left(I - \frac{z}{z - \xi_{n+1}} S \right) \varphi(z) \left(I + \frac{\xi_n}{z - \xi_n} Q \right), \quad (13)$$

where $Q = u_G v^T$ and $S = w v_R^T$, with v and w any vectors such that $u_G^T v = 1$ and $v_R^T w = 1$. From Theorems 8 and 9 we find that $\varphi_d(z) = z^{-1} A_{-1}^d + A_0^d - I + z A_1^d$, with matrix coefficients

$$\begin{aligned} A_{-1}^d &= A_{-1}(I - Q), \\ A_0^d &= A_0 + \xi_n A_1 Q + \xi_{n+1}^{-1} S A_{-1} - \xi_{n+1}^{-1} S A_{-1} Q \\ &= A_0 + \xi_n A_1 Q + \xi_{n+1}^{-1} S A_{-1} - \xi_n S A_1 Q \\ A_1^d &= (I - S) A_1. \end{aligned} \quad (14)$$

The two expressions for A_0^d coincide since, from (7), one has $A_1G = RA_{-1}$, and therefore $\xi_n v_R^T A_1 u_G = \xi_{n+1}^{-1} v_R^T A_{-1} u_G$.

From Theorems 8 and 9 it follows that the matrix polynomial $B_d(z) = z\varphi_d(z)$ has roots $0, \xi_1, \dots, \xi_{n-1}, \xi_{n+2}, \dots, \xi_{2n}, \infty$. In particular, $\varphi_d(z)$ is non-singular on the unit circle and on the annulus $|\xi_{n-1}| < |z| < |\xi_{n+2}|$.

4 Canonical factorizations

Consider the Laurent matrix polynomial $\varphi_s(z)$, for $s \in \{\mathbf{r}, \ell, \mathbf{d}\}$, where $\varphi_s(z)$ is obtained by applying one of the shift techniques described in Section 3. Our goal in this section is to show that $\varphi_s(z)$ and $\varphi_s(z^{-1})$ admit a (weak) canonical factorization, and to determine relations between G , R , \widehat{G} and \widehat{R} , and the solutions of the transformed equations

$$A_{-1}^s + (A_0^s - I)X + A_1^s X^2 = 0, \quad (15)$$

$$X^2 A_{-1}^s + X(A_0^s - I) + A_1^s = 0, \quad (16)$$

$$A_{-1}^s X^2 + (A_0^s - I)X + A_1^s = 0, \quad (17)$$

$$A_{-1}^s + X(A_0^s - I) + X^2 A_1^s = 0. \quad (18)$$

4.1 Shift to the right

Consider the function $\varphi_{\mathbf{r}}(z)$ obtained by shifting ξ_n to zero, defined in (10). Independently of the recurrent/transient case, the matrix Laurent polynomial $\varphi_{\mathbf{r}}(z)$ has a canonical factorization, as shown by the following theorem.

Theorem 10. *Define $Q = u_G v^T$, where v is any vector such that $u_G^T v = 1$. The function $\varphi_{\mathbf{r}}(z)$, defined in (10), has the factorization*

$$\varphi_{\mathbf{r}}(z) = (I - zR_{\mathbf{r}})K_{\mathbf{r}}(I - z^{-1}G_{\mathbf{r}}),$$

where $G_{\mathbf{r}} = G - \xi_n Q$, $R_{\mathbf{r}} = R$ and $K_{\mathbf{r}} = K$. This factorization is canonical in the positive recurrent case, and weakly canonical otherwise. Moreover, the eigenvalues of $G_{\mathbf{r}}$ are those of G , except for the eigenvalue ξ_n which is replaced by zero; the matrices $G_{\mathbf{r}}$ and $R_{\mathbf{r}}$ are the solutions with minimal spectral radius of the equations (15) and (16), respectively.

Proof. Since $GQ = \xi_n Q$, then $(I - z^{-1}G)(I + \frac{\xi_n}{z - \xi_n}Q) = I - z^{-1}(G - \xi_n Q)$. Hence, from (7) and (10), we find that

$$\varphi_{\mathbf{r}}(z) = (I - zR)K(I - z^{-1}G_{\mathbf{r}}), \quad G_{\mathbf{r}} = G - \xi_n Q,$$

which proves the factorization of $\varphi_{\mathbf{r}}(z)$. Since $\det(I - z^{-1}G_{\mathbf{r}}) = \frac{z}{z - \xi_n} \det(I - z^{-1}G)$, then the eigenvalues of $G_{\mathbf{r}}$ are the eigenvalues of G , except for the eigenvalue ξ_n which is replaced by 0. Thus the factorization is canonical in the positive recurrent case, weak canonical otherwise. A direct inspection shows

that $G_{\mathbf{r}}$ and R solve (15) and (16), respectively. They are the solutions with minimal spectral radius since their eigenvalues coincide with the n roots with smallest modulus of $B_{\mathbf{r}}(z)$ and of $zB_{\mathbf{r}}(z^{-1})$, respectively. \square

For the existence of the (weak) canonical factorization of $\varphi_{\mathbf{r}}(z^{-1})$ we distinguish the null recurrent from the non null recurrent case. In the latter case, since the matrix polynomial $B_{\mathbf{r}}(z)$ is still singular on the unit circle, the function $\varphi_{\mathbf{r}}(z^{-1})$ has a weak canonical factorization, as stated by the following theorem.

Theorem 11. *Assume that $\xi_n = \xi_{n+1} = 1$ (i.e., the QBD is null recurrent); define $Q = u_G v_G^T$, where $v_G^T u_G = 1$. Normalize $u_{\hat{R}}$ so that $v_G^T \hat{K}^{-1} u_{\hat{R}} = -1$. The function $\varphi_{\mathbf{r}}(z)$, defined in (10), has the weak canonical factorization*

$$\varphi_{\mathbf{r}}(z^{-1}) = (I - z\hat{R}_{\mathbf{r}})\hat{K}_{\mathbf{r}}(I - z^{-1}\hat{G}_{\mathbf{r}})$$

with

$$\hat{R}_{\mathbf{r}} = \hat{R} + u_{\hat{R}} v_G^T \hat{K}^{-1}, \quad (19)$$

$$\hat{K}_{\mathbf{r}} = \hat{K} - (u_{\hat{R}} + \hat{K} u_G) v_G^T, \quad (20)$$

$$\hat{G}_{\mathbf{r}} = \hat{G} + (u_G + \hat{K}^{-1} u_{\hat{R}}) v_G^T. \quad (21)$$

The eigenvalues of $\hat{R}_{\mathbf{r}}$ are those of \hat{R} , except for the eigenvalue 1 which is replaced by 0; the eigenvalues of $\hat{G}_{\mathbf{r}}$ are the same as the eigenvalues of \hat{G} . Moreover, the matrices $\hat{G}_{\mathbf{r}}$ and $\hat{R}_{\mathbf{r}}$ are the solutions of minimum spectral radius of (17) and (18), respectively.

Proof. Since both G and \hat{G} are stochastic, $u_G = u_{\hat{G}} = e$. As u_G and $v_{\hat{G}}$ are right and left eigenvector, respectively, of \hat{G} corresponding to the same eigenvalue, then $v_{\hat{G}}^T u_G \neq 0$ and we may scale the vectors in such a way that $v_{\hat{G}}^T u_G = 1$. In view of part 2 of Theorem 6 we have $v_{\hat{G}}^T \hat{K}^{-1} u_{\hat{R}} < 0$, so that we may normalize $u_{\hat{R}}$ so that $v_{\hat{G}}^T \hat{K}^{-1} u_{\hat{R}} = -1$. Observe that, for the matrix $\hat{R}_{\mathbf{r}}$ of (19), we have $\hat{R}_{\mathbf{r}} u_{\hat{R}} = u_{\hat{R}} + u_{\hat{R}} (v_{\hat{G}}^T \hat{K}^{-1} u_{\hat{R}}) = 0$. From this property, in view of Theorem 7, it follows that the eigenvalues of $\hat{R}_{\mathbf{r}}$ are those of \hat{R} , except for the eigenvalue 1, which is replaced by 0. Similarly, for the matrix $\hat{G}_{\mathbf{r}}$ of (21), one finds that $v_{\hat{G}}^T \hat{G}_{\mathbf{r}} = v_{\hat{G}}^T \hat{G} = v_{\hat{G}}^T$, therefore the matrix $\hat{G}_{\mathbf{r}}$ has the same eigenvalues of \hat{G} for Theorem 7. Now we prove that $\hat{R}_{\mathbf{r}}$ solves equation (18). By replacing X with $\hat{R}_{\mathbf{r}}$ and the block coefficients with the expressions in (11), the left hand side of equation (18) becomes $A_{-1}(I - Q) + \hat{R}_{\mathbf{r}}(A_0 - I + A_1 Q) + \hat{R}_{\mathbf{r}}^2 A_1$. Observe that $\hat{R}_{\mathbf{r}}^2 = \hat{R}^2 + u_{\hat{R}} v_{\hat{G}}^T \hat{K}^{-1} \hat{R}$. By replacing $\hat{R}_{\mathbf{r}}$ and $\hat{R}_{\mathbf{r}}^2$ with their expressions in terms of \hat{R} , and by using the property $A_{-1} + \hat{R} A_0 + \hat{R}^2 A_1 = \hat{R}$, the left hand

side of equation (18) becomes

$$\begin{aligned}
A_{-1}(I - Q) + \widehat{R}_r(A_0 - I + A_1Q) + \widehat{R}_r^2A_1 \\
&= -A_{-1}Q + \widehat{R}A_1Q + u_{\widehat{R}}v_{\widehat{G}}^T\widehat{K}^{-1}(A_0 - I + A_1Q) + u_{\widehat{R}}v_{\widehat{G}}^T\widehat{K}^{-1}\widehat{R}A_1 \\
&= -A_{-1}Q + A_{-1}\widehat{G}Q - u_{\widehat{R}}v_{\widehat{G}}^T\widehat{G}Q + u_{\widehat{R}}v_{\widehat{G}}^T\widehat{K}^{-1}(A_0 - I + \widehat{R}A_1) \\
&= -u_{\widehat{R}}v_{\widehat{G}}^TQ + u_{\widehat{R}}v_{\widehat{G}}^T = 0,
\end{aligned}$$

where the first equality holds since $\widehat{R}A_1 = A_{-1}\widehat{G}$ and $\widehat{K}^{-1}A_1 = -\widehat{G}$, the second and third equalities hold since $\widehat{G}Q = Q$, $\widehat{K} = A_0 - I + \widehat{R}A_1$ and $v_{\widehat{G}}^TQ = v_{\widehat{G}}^T$. In view of Theorem 2, where the role of G is replaced by R , the function $\varphi_r(z^{-1})$ has the desired weak canonical factorization where $\widehat{K}_r = A_0^r - I + \widehat{R}_rA_1^r$ and $\widehat{G}_r = -\widehat{K}_r^{-1}A_1^r$. To prove that \widehat{K}_r is given by (20), we replace the expression (19) of \widehat{R}_r in \widehat{K}_r and this yields

$$\begin{aligned}
\widehat{K}_r &= A_0 - I + A_1Q + (\widehat{R} + u_{\widehat{R}}v_{\widehat{G}}^T\widehat{K}^{-1})A_1 \\
&= A_0 - I + \widehat{R}A_1 + u_{\widehat{R}}v_{\widehat{G}}^T\widehat{K}^{-1}A_1 + A_1Q \\
&= \widehat{K} + (-u_{\widehat{R}} + A_1u_G)v_{\widehat{G}}^T = \widehat{K} - (u_{\widehat{R}} + \widehat{K}u_G)v_{\widehat{G}}^T.
\end{aligned}$$

Here we have used the properties $\widehat{K}^{-1}A_1 = -\widehat{G}$, $v_{\widehat{G}}^T\widehat{G} = v_{\widehat{G}}^T$, $\widehat{G}u_G = u_G$, and $\widehat{K} = A_0 - I + \widehat{R}A_1$. Finally, we prove that \widehat{G}_r is given by (21). By using the Sherman-Woodbury-Morrison formula we may write $\widehat{K}_r^{-1} = \widehat{K}^{-1} + \gamma\widehat{K}^{-1}(u_{\widehat{R}} + \widehat{K}u_G)v_{\widehat{G}}^T\widehat{K}^{-1}$, where $\gamma = 1/(1 - v_{\widehat{G}}^T\widehat{K}^{-1}(u_{\widehat{R}} + \widehat{K}u_G)) = 1$ for the assumptions on $v_{\widehat{G}}$, $u_{\widehat{R}}$ and u_G . Hence, $\widehat{K}_r^{-1} = \widehat{K}^{-1} + (\widehat{K}^{-1}u_{\widehat{R}} + u_G)v_{\widehat{G}}^T\widehat{K}^{-1}$ so that $\widehat{G}_r = -\widehat{K}_r^{-1}A_1^r = \widehat{G} + (u_G + \widehat{K}^{-1}u_{\widehat{R}})v_{\widehat{G}}^T$. \square

In the non null recurrent case, the function $\varphi_r(z^{-1})$ has a (weak) canonical factorization, as stated by the following theorem.

Theorem 12. *Assume that $\xi_n < \xi_{n+1}$ (i.e., the QBD is not null recurrent). Define $Q = u_Gv^T$, with v any vector such that $u_G^Tv = 1$ and $\xi_nv^T\widehat{G}u_G \neq 1$. The Laurent matrix polynomial $\varphi_r(z^{-1})$ defined in (10), has the factorization*

$$\varphi_r(z^{-1}) = (I - z\widehat{R}_r)\widehat{K}_r(I - z^{-1}\widehat{G}_r),$$

where

$$\begin{aligned}
W_r &= W - \xi_nQWR, \\
\widehat{K}_r &= A_0^r - I + A_{-1}^r\widehat{G}_r = A_0^r - I + \widehat{R}_rA_1, \\
G_r &= G - \xi_nQ, \\
\widehat{G}_r &= W_rRW_r^{-1}, \\
\widehat{R}_r &= W_r^{-1}G_rW_r.
\end{aligned}$$

Moreover, $\widehat{G}_{\mathbf{r}}$ and $\widehat{R}_{\mathbf{r}}$ are the solutions with minimal spectral radius of (17) and (18), respectively. The factorization is canonical if $\xi_n = 1$ and weakly canonical if $\xi_{n+1} = 1$.

Proof. As a first step, we show that the matrix $W_{\mathbf{r}} = \sum_{i=0}^{+\infty} G_{\mathbf{r}}^i K^{-1} R^i$, with $G_{\mathbf{r}} = G - \xi_n Q$, is nonsingular, so that we can apply property 3 of Theorem 1 to the matrix Laurent polynomial $\varphi_{\mathbf{r}}(z)$ of Theorem 8. Observe that $G_{\mathbf{r}}^i = G^i - \xi_n Q G^{i-1}$, for $i \geq 1$. Therefore, we may write

$$\begin{aligned} W_{\mathbf{r}} &= K^{-1} + \sum_{i=1}^{+\infty} (G^i - \xi_n Q G^{i-1}) K^{-1} R^i \\ &= K^{-1} + \sum_{i=1}^{+\infty} G^i K^{-1} R^i - \xi_n Q \left(\sum_{i=0}^{+\infty} G^i K^{-1} R^i \right) R = W - \xi_n Q W R. \end{aligned}$$

Since $\det W \neq 0$ by Theorem 2, part 3, then $\det W_{\mathbf{r}} = \det(I - \xi_n Q W R W^{-1}) \det W$. Moreover, since $Q = u_G v^T$, then the matrix $I - \xi_n Q W R W^{-1}$ is nonsingular if and only if $\xi_n v^T W R W^{-1} u_G \neq 1$. Since $\widehat{G} = W R W^{-1}$, the latter condition holds if $\xi_n v^T \widehat{G} u_G \neq 1$, which we assume, and so, the matrix $W_{\mathbf{r}}$ is nonsingular. If $\xi_n = 1$, since $\rho(G_{\mathbf{r}}) < 1$ and $\rho(R) < 1$, from 3 of Theorem 1 applied to the matrix Laurent polynomial $\varphi_{\mathbf{r}}(z)$, we deduce that $\varphi_{\mathbf{r}}(z^{-1})$ has the canonical factorization $\varphi_{\mathbf{r}}(z^{-1}) = (I - z \widehat{R}_{\mathbf{r}}) \widehat{K}_{\mathbf{r}} (I - z^{-1} \widehat{G}_{\mathbf{r}})$ with $\widehat{K}_{\mathbf{r}} = A_0^{\mathbf{r}} - I + A_{-1}^{\mathbf{r}} \widehat{G}_{\mathbf{r}} = A_0^{\mathbf{r}} - I + \widehat{R}_{\mathbf{r}} A_1$ and $\widehat{G}_{\mathbf{r}} = W_{\mathbf{r}} R W_{\mathbf{r}}^{-1}$, $\widehat{R}_{\mathbf{r}} = W_{\mathbf{r}}^{-1} G_{\mathbf{r}} W_{\mathbf{r}}$. If $\xi_{n+1} = 1$, we can apply the above property to the function $\varphi^{(t)}(z) = \varphi(tz)$ with $\xi_n < t < 1$ and obtain the canonical factorization for $\varphi^{(t)}(z)$. Scaling again the variable z by t^{-1} we obtain a weak canonical factorization for $\varphi(z)$. With the same arguments used in the proof of Theorem 10, we may prove that $\widehat{G}_{\mathbf{r}}$ and $\widehat{R}_{\mathbf{r}}$ are the solutions with minimal spectral radius of (17) and (18). \square

In the above theorem we can choose $v = v_{\widehat{G}}$, so that $v_{\widehat{G}}^T \widehat{G} = \xi_{n+1}^{-1} v_{\widehat{G}}^T$. Since $u_G > 0$, then $v_{\widehat{G}}^T u_G > 0$ and we can normalize the vectors so that $v_{\widehat{G}}^T u_G = 1$. In this way we obtain $\xi_n v_{\widehat{G}}^T \widehat{G} u_G = \xi_n \xi_{n+1}^{-1} v_{\widehat{G}}^T u_G = \xi_n \xi_{n+1}^{-1} < 1$. Therefore, the assumption on v of Theorem 12 is satisfied.

4.2 Shift to the left

As for the right shift, the matrix Laurent polynomial $\varphi_{\ell}(z)$ defined by (12) and obtained by shifting ξ_{n+1} to infinity, has a canonical factorization, as shown by the following theorem.

Theorem 13. Define $S = w v_R^T$, where w is any vector such that $v_R^T w = 1$. The function $\varphi_{\ell}(z)$ defined in (12), has the factorization

$$\varphi_{\ell}(z) = (I - z R_{\ell}) K_{\ell} (I - z^{-1} G_{\ell}),$$

where $R_{\ell} = R - \xi_{n+1}^{-1} S$, $G_{\ell} = G$ and $K_{\ell} = K$. This factorization is canonical in the transient case, weakly canonical otherwise. Moreover, the eigenvalues of

R_ℓ are those of R , except for the eigenvalue ξ_{n+1}^{-1} which is replaced by zero; the matrices G and R_ℓ are the solutions with minimal spectral radius of equations (15) and (16), respectively.

Proof. The proof can be carried out as the proof of Theorem 10, after observing that $(I - \frac{z}{z - \xi_{n+1}}S)(I - zR) = I - z(R - \xi_{n+1}^{-1}S)$. \square

Similarly to the shift to the right, we may prove the following results concerning the canonical factorization of $\varphi_\ell(z^{-1})$:

Theorem 14. Assume that $\xi_n = \xi_{n+1} = 1$ (i.e., the QBD is null recurrent). Define $S = u_{\hat{R}}v_R^T$, where $v_R^T u_{\hat{R}} = 1$. Normalize $v_{\hat{G}}$ such that $v_{\hat{G}}^T \hat{K}^{-1} u_{\hat{R}} = -1$. The function $\varphi_\ell(z)$ defined in (12) has the weak canonical factorization

$$\varphi_\ell(z^{-1}) = (I - z\hat{R}_\ell)\hat{K}_\ell(I - z^{-1}\hat{G}_\ell)$$

with

$$\begin{aligned}\hat{R}_\ell &= \hat{R} + u_{\hat{R}}(v_R^T + v_{\hat{G}}^T \hat{K}^{-1}), \\ \hat{K}_\ell &= \hat{K} - u_{\hat{R}}(v_{\hat{G}}^T + v_R^T \hat{K}), \\ \hat{G}_\ell &= \hat{G} + \hat{K}^{-1} u_{\hat{R}} v_{\hat{G}}^T.\end{aligned}$$

The eigenvalues of \hat{G}_ℓ are those of \hat{G} , except for the eigenvalue 1 which is replaced by 0; the eigenvalues of \hat{R}_ℓ are the same as the eigenvalues of \hat{R} . Moreover, the matrices \hat{G}_ℓ and \hat{R}_ℓ are the solutions of minimum spectral radius of equations (17) and (18), respectively.

If $\xi_n < \xi_{n+1}$ we have the following result.

Theorem 15. Assume that $\xi_n < \xi_{n+1}$ (i.e., the QBD is not null recurrent). Define $Q = wv_R^T$, with w any vector such that $v_R^T w = 1$ and $\xi_{n+1}^{-1} v_R^T \hat{R} w \neq 1$. The Laurent matrix polynomial $\varphi_\ell(z^{-1})$ defined in (12), has the factorization

$$\varphi_\ell(z^{-1}) = (I - z\hat{R}_\ell)\hat{K}_\ell(I - z^{-1}\hat{G}_\ell)$$

with

$$\begin{aligned}\hat{G}_\ell &= W_\ell^{-1} R_\ell W_\ell, \\ W_\ell &= W - \xi_{n+1}^{-1} G W S, \\ R_\ell &= R - \xi_{n+1}^{-1} S, \\ \hat{R}_\ell &= W_\ell G W_\ell^{-1}, \\ \hat{K}_\ell &= A_0^\ell - I + A_{-1}^\ell \hat{G}_\ell = A_0^\ell - I + \hat{R}_\ell A_1.\end{aligned}$$

Moreover, \hat{G}_ℓ and \hat{R}_ℓ are the solutions with minimal spectral radius of equations (17) and (18), respectively. The factorization is canonical if $\xi_n < 1$, is weakly canonical if $\xi_n = 1$.

4.3 Double shift

Consider the matrix function $\varphi_{\mathbf{d}}(z)$ defined in (13), obtained by shifting ξ_n to 0 and ξ_{n+1} to ∞ . The matrix Laurent polynomial $\varphi_{\mathbf{d}}(z)$, has a canonical factorization, as shown by the following theorem.

Theorem 16. *Define $Q = u_G v^T$ and $S = w v_R^T$, with v and w any vectors such that $u_G^T v = 1$ and $v_R^T w = 1$. The function $\varphi_{\mathbf{d}}(z)$ defined in (13), has the following canonical factorization*

$$\varphi_{\mathbf{d}}(z) = (I - zR_{\mathbf{d}})K_{\mathbf{d}}(I - z^{-1}G_{\mathbf{d}}),$$

where $R_{\mathbf{d}} = R - \xi_{n+1}^{-1}S$, $G_{\mathbf{d}} = G - \xi_n Q$ and $K_{\mathbf{d}} = K$. Moreover, $G_{\mathbf{d}}$ and $R_{\mathbf{d}}$ are the solutions with minimal spectral radius of equations (15) and (16), respectively.

Proof. The proof can be carried out as the proof of Theorems 10 and 13, since $(I - z^{-1}G)(I + \frac{\xi_n}{z - \xi_n}Q) = I - z^{-1}(G - \xi_n Q)$ and $(I - \frac{z}{z - \xi_{n+1}}S)(I - zR) = I - z(R - \xi_{n+1}^{-1}S)$. \square

We show that in the null recurrent case, where $\xi_n = \xi_{n+1} = 1$, the matrix Laurent polynomial $\varphi_{\mathbf{d}}(z^{-1})$ has also a canonical factorization:

Theorem 17. *Assume that $\xi_n = \xi_{n+1} = 1$. Define $Q = u_G v_G^T$ and $S = u_{\hat{R}} v_{\hat{R}}^T$, with $u_G^T v_G = 1$ and $v_{\hat{R}}^T u_{\hat{R}} = 1$. Normalize the vectors $v_{\hat{G}}$ and $u_{\hat{R}}$ so that $v_{\hat{G}}^T \hat{K}^{-1} u_{\hat{R}} = -1$. The function $\varphi_{\mathbf{d}}(z^{-1})$ defined in (13), has the following canonical factorization*

$$\varphi_{\mathbf{d}}(z^{-1}) = (I - z\hat{R}_{\mathbf{d}})\hat{K}_{\mathbf{d}}(I - z^{-1}\hat{G}_{\mathbf{d}})$$

where

$$\begin{aligned}\hat{R}_{\mathbf{d}} &= \hat{R} + u_{\hat{R}} v_{\hat{G}}^T \hat{K}^{-1}, \\ \hat{G}_{\mathbf{d}} &= \hat{G} + \hat{K}^{-1} u_{\hat{R}} v_{\hat{G}}^T, \\ \hat{K} &= A_{-1} \hat{G} + A_0 - I, \\ \hat{K}_{\mathbf{d}} &= \hat{K} - u_{\hat{R}} v_{\hat{G}}^T.\end{aligned}$$

Moreover, the matrices $\hat{G}_{\mathbf{d}}$ and $\hat{R}_{\mathbf{d}}$ are the solutions with minimal spectral radius of equations (17) and (18), respectively.

Proof. In view of part 2 of Theorem 6, we have $v_{\hat{G}}^T \hat{K}^{-1} u_{\hat{R}} < 0$, therefore we may normalize the vectors so that $v_{\hat{G}}^T \hat{K}^{-1} u_{\hat{R}} = -1$. Observe that, for the matrix $\hat{G}_{\mathbf{d}}$ defined in the theorem, we have

$$Q\hat{G}_{\mathbf{d}} = u_G v_G^T (\hat{G} + \hat{K}^{-1} u_{\hat{R}} v_{\hat{G}}^T) = u_G (v_G^T + (v_G^T \hat{K}^{-1} u_{\hat{R}}) v_{\hat{G}}^T) = 0.$$

Similarly, one has $\hat{R}_{\mathbf{d}} S = 0$. From Theorem 7, it follows that the eigenvalues of $\hat{G}_{\mathbf{d}}$ are those of \hat{G} , except for the eigenvalue 1, which is replaced by 0; the same

holds for \widehat{R}_d . Now we prove that \widehat{G}_d solves the equation (17). By replacing X with \widehat{G}_d and the block coefficients with the expressions in (14), the left hand side of the quadratic equation (17) becomes

$$A_{-1}(I - Q)\widehat{G}_d^2 + (A_0 - I + (I - S)A_1Q + SA_{-1})\widehat{G}_d + (I - S)A_1.$$

Since $Q\widehat{G}_d = 0$, the above expression simplifies to

$$A_{-1}\widehat{G}_d^2 + (A_0 - I + SA_{-1})\widehat{G}_d + (I - S)A_1.$$

Observe that $\widehat{G}_d^2 = \widehat{G}^2 + \widehat{G}\widehat{K}^{-1}u_{\widehat{R}}v_{\widehat{G}}^T$. By replacing \widehat{G}_d and \widehat{G}_d^2 with their expressions in terms of \widehat{G} , and by using the property $A_{-1}\widehat{G}^2 + (A_0 - I)\widehat{G} + A_1 = 0$, we get

$$\begin{aligned} & A_{-1}\widehat{G}_d^2 + (A_0 - I + SA_{-1})\widehat{G}_d + (I - S)A_1 = \\ & A_{-1}\widehat{G}\widehat{K}^{-1}u_{\widehat{R}}v_{\widehat{G}}^T + (A_0 - I)\widehat{K}^{-1}u_{\widehat{R}}v_{\widehat{G}}^T + SA_{-1}\widehat{G} + SA_{-1}\widehat{K}^{-1}u_{\widehat{R}}v_{\widehat{G}}^T - SA_1 = \\ & (A_{-1}\widehat{G} + A_0 - I)\widehat{K}^{-1}u_{\widehat{R}}v_{\widehat{G}}^T + SA_{-1}\widehat{G} + SA_{-1}\widehat{K}^{-1}u_{\widehat{R}}v_{\widehat{G}}^T - SA_1 + \widehat{K}^{-1}u_{\widehat{R}}v_{\widehat{G}}^T = \\ & u_{\widehat{R}}v_{\widehat{G}}^T + SA_{-1}\widehat{G} + SA_{-1}\widehat{K}^{-1}u_{\widehat{R}}v_{\widehat{G}}^T - SA_1. \end{aligned}$$

This latter equation is zero. Indeed, $-SA_{-1}\widehat{K}^{-1} = S\widehat{R} = S$ and $Su_{\widehat{R}} = u_{\widehat{R}}$, therefore $SA_{-1}\widehat{K}^{-1}u_{\widehat{R}}v_{\widehat{G}}^T = -u_{\widehat{R}}v_{\widehat{G}}^T$; moreover, $SA_{-1}\widehat{G} = S\widehat{R}A_1 = SA_1$. Similarly, we may prove that \widehat{R}_d solves the equation $A_{-1}^d\widehat{R}_d + XA_0^d - I + X^2A_1^d = 0$. Since \widehat{G}_d and \widehat{R}_d are the solutions of minimal spectral radius of equations (17) and (18), we may apply Theorem 3.20 of [3] and conclude that $\varphi_d(z^{-1})$ has desired the canonical factorization with

$$\begin{aligned} \widehat{K}_d &= A_{-1}^d\widehat{G}_d + A_0^d - I = A_{-1}^d(\widehat{G} + \widehat{K}^{-1}u_{\widehat{R}}v_{\widehat{G}}^T) + A_0^d - I \\ &= \widehat{K} + A_{-1}^d\widehat{K}^{-1}u_{\widehat{R}}v_{\widehat{G}}^T = \widehat{K} - u_{\widehat{R}}v_{\widehat{G}}^T, \end{aligned}$$

since $A_{-1}^d\widehat{K}^{-1} = -\widehat{R}$. □

References

- [1] A. Berman and R. J. Plemmons. *Nonnegative matrices in the mathematical sciences*, volume 9 of *Classics in Applied Mathematics*. SIAM, Philadelphia, PA, 1994.
- [2] D. A. Bini, S. Dendievel, G. Latouche, and B. Meini. General solution of the Poisson equation for QBDs. In preparation, 2016.
- [3] D. A. Bini, G. Latouche, and B. Meini. *Numerical methods for structured Markov chains*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2005.

- [4] D. A. Bini, B. Meini, and I. M. Spitkovsky. Shift techniques and canonical factorizations in the solution of M/G/1-type Markov chains. *Stochastic Models*, 21(2-3):279–302, 2005.
- [5] A. Brauer. Limits for the characteristic roots of a matrix. IV. Applications to stochastic matrices. *Duke Math. J.*, 19:75–91, 1952.
- [6] K. F. Clancey and I. Gohberg. *Factorization of matrix functions and singular integral operators*, volume 3 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel-Boston, Mass., 1981.
- [7] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials*. Academic Press Inc., New York, 1982. Computer Science and Applied Mathematics.
- [8] C.-H. Guo. Comments on a shifted cyclic reduction algorithm for quasi-birth-death problems. *SIAM J. Matrix Anal. Appl.*, 24(4):1161–1166, 2003.
- [9] C.-H. Guo and N. J. Higham. Iterative solution of a nonsymmetric algebraic Riccati equation. *SIAM J. Matrix Anal. Appl.*, 29(2):396–412, 2007.
- [10] C.-H. Guo, B. Iannazzo, and B. Meini. On the doubling algorithm for a (shifted) nonsymmetric algebraic Riccati equation. *SIAM J. Matrix Anal. Appl.*, 29(4):1083–1100, 2007.
- [11] C. He, B. Meini, and N. H. Rhee. A shifted cyclic reduction algorithm for quasi-birth-death problems. *SIAM J. Matrix Anal. Appl.*, 23(3):673–691, 2001/02.
- [12] N. J. Higham. *Functions of Matrices: Theory and Computation*. SIAM, Philadelphia, PA, USA, 2008.
- [13] N. J. Higham and H.-M. Kim. Numerical analysis of a quadratic matrix equation. *IMA J. Numer. Anal.*, 20(4):499–519, 2000.
- [14] B. Iannazzo and F. Poloni. A subspace shift technique for nonsymmetric algebraic Riccati equations associated with an M-matrix. *Numer. Linear Algebra Appl.*, 20(3):440–452, 2013.
- [15] G. Latouche and V. Ramaswami. *Introduction to matrix analytic methods in stochastic modeling*. ASA-SIAM Series on Statistics and Applied Probability. SIAM, Philadelphia, PA, 1999.
- [16] B. Meini. A shift-and-deflate technique for quadratic matrix polynomials. *Linear Algebra and its Applications*, 438(4):1946 – 1961, 2013.
- [17] M. F. Neuts. *Matrix-geometric solutions in stochastic models*, volume 2 of *Johns Hopkins Series in the Mathematical Sciences*. Johns Hopkins University Press, Baltimore, Md., 1981. An algorithmic approach.
- [18] F. Tisseur and K. Meerbergen. The quadratic eigenvalue problem. *SIAM Rev.*, 43(2):235–286, 2001.

Appendix

Here we provide the proof of part 2 of Theorem 6, i.e., $v_G^T K^{-1} u_R < 0$ and $v_{\hat{G}}^T \hat{K}^{-1} u_{\hat{R}} < 0$. The proof is based on an argument of accessibility for the states of the doubly infinite QBD. Assumptions 3 and 4, together imply that $\mathcal{S}_* = \mathcal{S}$ and so we have the following property.

A.1 For any $i, j \in \mathcal{S}$, for any level k and k' , there is a path from (k, i) to (k', j) .

We prove that $v_G^T K^{-1} u_R < 0$; the proof that $v_{\hat{G}}^T \hat{K}^{-1} u_{\hat{R}} < 0$ is similar and is left to the reader. For the sake of simplicity we write u in place of u_R and v in place of v_G . The proof consists in analyzing the sign properties of the components of the vectors u and $-K^{-1}u$, and relies on the irreducibility assumptions.

The vector u

Case 1. R is irreducible. Then $u > 0$, and $K^{-1}u < 0$ also, so that $v^T K^{-1}u < 0$.

Case 2. R is reducible. We need to define various subsets of the set of phases $\mathcal{S} = \{1, \dots, n\}$. The most important ones define a partition of \mathcal{S} into the four subsets $\mathcal{S}_1, \tilde{\mathcal{S}}_1, \tilde{\mathcal{S}}_b$ and \mathcal{S}_a that we define later. As we assume that the matrix A is irreducible, we have after a suitable permutation of rows and columns $\begin{bmatrix} R_1 & R_{12} \\ 0 & R_2 \end{bmatrix}$, where R_1 is irreducible and R_2 is strictly upper-triangular. The proof is in [15, Theorem 7.2.2, page 154]. The rows and columns of R_1 are indexed by \mathcal{S}_1 and those of R_2 are indexed by \mathcal{S}_2 , and so $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$. By Assumption 4, \mathcal{S}_1 is not empty.

Concerning the eigenvector u of R , we have : $u_i > 0$ for any i in \mathcal{S}_1 and $u_i = 0$ for i in \mathcal{S}_2 . The physical meaning of the partition $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ is given below. Consider the doubly infinite QBD process.

B.1. For any $i \in \mathcal{S}_1$, for any level k , for any displacement $h \geq 1$, there exists $i' \in \mathcal{S}_1$ such that there is a path from (k, i) to $(k+h, i')$ avoiding level k and the levels below.

B.2. For any $i \in \mathcal{S}_2$, for any level k , for any displacement $h \geq 1$, for any $i' \in \mathcal{S}_1$, there is no path from (k, i) to $(k+h, i')$ avoiding level k ; i.e., any path from (k, i) to $(k+h, i')$ has to go through level k or $k-1$.

The vector $K^{-1}u$

The matrix $-K^{-1}$ is about transitions within a level k without visiting level $k-1$: $(-K^{-1})_{ij}$ is the expected number of visits to (k, j) , starting from (k, i) before any visit to level $k-1$. Clearly,

C.1. $(-K^{-1})_{ij} > 0$ if and only if there exists a path from (k, i) to (k, j) that avoids level $k-1$, possibly after visiting some states in level $k+1$ or above.

Define $w = -K^{-1}u$. We have $w \geq u$, so that $w_i > 0$ for all i in \mathcal{S}_1 . Furthermore, there may be some phases i in \mathcal{S}_2 such that $w_i > 0$; define $\tilde{\mathcal{S}}_1$ as the subset of phases i such that $u_i = 0, w_i > 0$, and define $\tilde{\mathcal{S}}_2 = \mathcal{S}_2 \setminus \tilde{\mathcal{S}}_1$, thus $\mathcal{S}_2 = \tilde{\mathcal{S}}_1 \cup \tilde{\mathcal{S}}_2$.

Case 1. $w_i > 0$ for all i , i.e., $\tilde{\mathcal{S}}_2$ is empty. Then, $v^T w > 0$ and we are done.

Case 2. The set $\tilde{\mathcal{S}}_2$ is not empty, there are phases i such that $w_i = 0$. From B.1, B.2 and C.1, we find that the physical meaning is as follows.

D.1. For any $i \in \tilde{\mathcal{S}}_1$, for any level k , for any displacement $h \geq 1$, there exists $i' \in \mathcal{S}_1$ such that there is a path from (k, i) to $(k + h, i')$ avoiding level $k - 1$ and any level below.

D.2. For any phase $i \in \tilde{\mathcal{S}}_2$, for any level k , for any displacement $h \geq 0$, for any $i' \in \mathcal{S}_1$, there is no path from (k, i) to $(k + h, i')$ avoiding level $k - 1$.

The product $v^T K^{-1} u$

By [15, Theorem 7.2.1, page 152], we have after a suitable permutation of rows and columns $G = \begin{bmatrix} G_a & 0 \\ G_{ba} & G_b \end{bmatrix}$ where G_a is irreducible and G_b is strictly lower triangular. The rows and columns of G_a are indexed by \mathcal{S}_a and those of G_b are indexed by \mathcal{S}_b ; define $n_b = |\mathcal{S}_b|$, thus, $\mathcal{S} = \mathcal{S}_a \cup \mathcal{S}_b$. The left eigenvector v of G is such that $v_i > 0$ for i in \mathcal{S}_a and $v_i = 0$ for i in \mathcal{S}_b .

By Assumption 4, \mathcal{S}_a is not empty. The physical interpretation is as follows.

E.1. For all $i \in \mathcal{S}_a$, there is $i' \in \mathcal{S}_a$ such that there is a path from (k, i) to $(k - 1, i')$, avoiding level $k - 1$ at the intermediary steps, independently of k .

E.2. For all $i \in \mathcal{S}_a$, for all $i' \in \mathcal{S}_b$, there is no path from (k, i) to $(k - 1, i')$, avoiding level $k - 1$ at the intermediary steps, independently of k .

Now, let us assume that $v^T K^{-1} u = 0$. This implies that if $v_i > 0$, then $w_i = 0$, so that $\mathcal{S}_a \subseteq \tilde{\mathcal{S}}_2$, with the possibility that $\tilde{\mathcal{S}}_b$, defined as $\tilde{\mathcal{S}}_b = \mathcal{S}_b \setminus (\mathcal{S}_1 \cup \tilde{\mathcal{S}}_1) = \tilde{\mathcal{S}}_2 \setminus \mathcal{S}_a$, may be empty or not empty. In summary, we have the table

	\mathcal{S}_1	$\tilde{\mathcal{S}}_1$	$\tilde{\mathcal{S}}_b$	\mathcal{S}_a
u	$u_i > 0$	$u_i = 0$	$u_i = 0$	$u_i = 0$
w	$w_i > 0$	$w_i > 0$	$w_i = 0$	$w_i = 0$
v	$v_i = 0$	$v_i = 0$	$v_i = 0$	$v_i > 0$

where \mathcal{S}_1 and \mathcal{S}_a are not empty, and we have $\mathcal{S}_2 = \tilde{\mathcal{S}}_1 \cup \tilde{\mathcal{S}}_b \cup \mathcal{S}_a$, $\tilde{\mathcal{S}}_2 = \tilde{\mathcal{S}}_b \cup \mathcal{S}_a$, $\mathcal{S}_b = \mathcal{S}_1 \cup \tilde{\mathcal{S}}_1 \cup \tilde{\mathcal{S}}_b$.

Now, let us fix some arbitrary initial level k_0 and take any phase i in \mathcal{S}_a . Since $\mathcal{S}_a \subset \tilde{\mathcal{S}}_2$, we know by D.2 that any path from (k, i) to any state $(k + h, j)$ with $h \geq 0$ and j in \mathcal{S}_1 must pass through level $k - 1$. By E.1 and E.2, from the state (k, i) , the first state $(k - 1, i')$ on any path through level $k - 1$ is such that i' is in \mathcal{S}_a . Therefore, we know that any path from (k, i) to any state $(k - 1 + h, j)$ with $h \geq 0$ and j in \mathcal{S}_1 must pass through level $k - 2$. We repeat the argument, and find that there is no path from $\mathbb{Z} \times \mathcal{S}_a$ to any state in $\mathbb{Z} \times \mathcal{S}_1$. This contradicts the property A.1.